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Invariant Subspaces of L^∞ of Certain Homogeneous Spaces

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It is proved that $L^\infty(G/H)$ does not contain any proper G-invariant closed subspaces of finite codimension, where the hypotheses are that G is a locally compact group, H a closed subgroup such that G/H is compact, and such that $\Delta_G|_H \neq \Delta_H$, where Δ denotes the modular function of the group involved.

THEOREM 1. *Let G be a locally compact group and let H be a closed subgroup such that G/H is compact. Suppose there is a $p_0 \in H$ for which $\Delta_H(p_0) \neq \Delta_G(p_0)$, where Δ_H and Δ_G are the modular functions of H and G, respectively. Suppose W is a closed subspace of (complex) $L^\infty(G/H)$ that is invariant under the natural action of G on $L^\infty(G/H)$ and such that W has finite codimension in $L^\infty(G/H)$. Then $W = L^\infty(G/H)$.*

This theorem overlaps a result [5, Theorem 1] of Rubel and Shields which considers invariant subspaces of $L^\infty(T)$, where $T = \{z: |z| = 1\}$ is the unit circle, under the action of the Möbius group M of all maps

$$\mu : z \mapsto e^{i\lambda} \frac{z - z_0}{1 - \bar{z}_0 z}, \lambda \in \mathbf{R}, |z_0| < 1.$$

Here, $G = M$ is isomorphic to $PSL(2, \mathbf{R})$, and is unimodular because it is simple. Let H be the subgroup of all elements of G that leave the point $z = 1$ fixed. Because G is transitive, we see that G/H may be taken as T, and G acts on G/H the way M acts on T. Now H is isomorphic to the group of matrices of the form,

$$m = \begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix}; \qquad a > 0, \quad b \text{ real},$$

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so that [1, Chapter 7, p. 84] $\Delta_H(m) = a^2$. Hence the considerations of [5, Theorem 1] satisfy the hypotheses of our Theorem 1. However [5] considers a large class of subspaces of $L^\infty(T)$, while we restrict our attention to $L^\infty(T)$ itself.

Our theorem is closely related to a theorem of Weil [6, p. 45] on the existence of invariant measures; where it is proved that the condition that Δ_G coincide with Δ_H on H is necessary and sufficient that G/H possess an invariant countably additive Borel measure. Our theorem may be considered as a generalization to the case of finitely additive measures, when G/H is compact. However when G/H is not compact, there may exist finitely additive invariant means even when countably additive ones do not exist. For example, when G is solvable, it is known [2, Theorem 1.2.1, p. 5; Theorem 1.2.6, p. 8; Theorem 2.2.1, p. 26] to be amenable, so that if H is a closed subgroup, then G/H admits an invariant mean.

To say that a locally compact group is amenable is to say that there exists an invariant mean on $L^\infty(G)$. For example, let

$$G = \left\{ \begin{pmatrix} x & y \\ 0 & 1 \end{pmatrix} : x > 0,\ y \text{ real} \right\},$$

$$H = \left\{ \begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix} : x > 0 \right\}.$$

Then G is solvable, $\Delta_G|_H \neq \Delta_H$ and G/H admits an invariant mean. To show that we need $\Delta_G|_H \neq \Delta_H$, take $G = \mathbf{R}$ and $H = \mathbf{Z}$ so that $G/H = T$, where G acts via rotation. Any character of T gives rise to an invariant subspace of codimension 1. In [5, Theorem 2], it was shown that there are actually invariant subspaces of codimension 1 in $L^\infty(T)$ that contain all the continuous functions. So the conditions on the modular functions and on the compactness of G/H are each needed.

After we proved Theorem 1, H. Furstenberg has found a different proof.

Proof of the Theorem. We suppose here only that G is a locally compact group and that H is a closed subgroup of G. By $C_c(G)$ we denote the space of all continuous complex-valued functions on G with compact support. The notions $C_c(G/H)$ and $L_c^\infty(G/H)$ are similarly used to denote compact support of the functions involved. If μ is a linear functional on $C_c(G/H)$ then $r(p)\mu$, for $p \in H$, means the right translate of μ by p, i.e.,

$$(r(p)\mu)(f) = \mu(r(p^{-1})f).$$

For $x \in G$ and μ as before, we mean by $x \cdot \mu$ the functional

$$(x \cdot \mu)(f) = \mu(f_x),$$

where

$$f_x(t) = f(x^{-1}t).$$

LEMMA 1. *Let $\mu_1, \mu_2, \ldots, \mu_N$ be nonnegative linear functionals on $C_c(G)$ such that*

(i) $r(p)\,\mu_i = \Delta_H(p)\,\mu_i$, $i = 1, 2, \ldots, N$,

(ii) $x \cdot \mu_i \leqslant M \sum_{j=1}^{n} \mu_j$, *for all* $x \in G$,

where M is a constant. If $\Delta_H(p_0) \neq \Delta_G(p_0)$ for some $p_0 \in H$, then $\mu_i = 0$ for all $i = 1, 2, \ldots, N$.

Proof. Choose such a p_0 and let ω be any compact neighborhood in G of p_0. Let G_0 be the subgroup of G generated by ω, and let $H_0 = G_0 \cap H$. Then G_0 and H_0 are open (and hence closed) subgroups of G and H, respectively. So $\Delta_G = \Delta_{G_0}$ on G_0 and $\Delta_H = \Delta_{H_0}$ on H_0. Note that G_0 is σ-compact. The restrictions of the μ_i to $C_c(G_0)$ satisfy the conditions (i) and (ii). Since we may take ω to contain any given compact set, the result is proved if we can prove it for G_0. In short, we may assume without loss of generality that G is σ-compact, and we do so. Let $\lambda = \sum \mu_i$. Then λ is finite, and for $M' = NM$ we have

$$x \cdot \lambda \leqslant M'\lambda.$$

So by a theorem of Mackey [3, Theorem 1.1, p. 106], λ is absolutely continuous with respect to Haar measure on G, since the null space of λ is translation invariant. Hence by the Radon–Nikodym theorem [4, p. 238] there exists a locally summable function ζ on G such that

$$\lambda(f) = \int \zeta(x) f(x)\, dx,$$

for $f \in C_c(G)$. The relation $x \cdot \lambda \leqslant M'\lambda$ implies that for each x, $\zeta(x^{-1}y) \leqslant M'\zeta(y)$ for almost all y. So by Tonelli's theorem [4, p. 270], there exists $y_0 \in G$ such that $\zeta(x^{-1}y_0) \leqslant M'\zeta(y_0)$ for almost all x. Thus $\| \zeta \|_\infty < \infty$. Relation (i) now implies that,

$$\| \zeta \|_\infty = (\Delta_H(p_0)/\Delta_G(p_0)) \| \zeta \|_\infty, \qquad \text{so that} \quad \| \zeta \|_\infty = 0.$$

Hence $\lambda = 0$ and so each $\mu_i = 0$.

LEMMA 2. *Let $\lambda_1, \lambda_2, \ldots, \lambda_N$ be a finite number of nonnegative linear functionals on $C_c(G/H)$ such that*

$$x \cdot \lambda_i \leqslant M \sum_{j=1}^{N} \lambda_j,$$

for all $x \in G$. If, for some $p_0 \in H$,

$$\Delta_H(p_0) \neq \Delta_G(p_0),$$

then $\lambda_j = 0$ for all j.

Proof. Let μ_i be the linear functionals defined on $C_c(G)$ by

$$\mu_i(f) = \lambda_i(\bar{f}),$$

where

$$\bar{f}(xH) = \int_H f(xh)\, d_l h.$$

Now [6, p. 43] $f \mapsto \bar{f}$ is a linear and onto map of $C_c(G)$ onto $C_c(G/H)$, so $\lambda_i = 0$ if $\mu_i = 0$. But the μ_i satisfy the hypotheses of Lemma 1, and we are done.

Proof of the Theorem (Completed). Let $W^\perp$ be the space of continuous linear functionals on $L^\infty(G/H)$ that annihilate W. Let $\nu_1, \nu_2, \ldots, \nu_N$ be a basis for $W^\perp$. The ν_i are continuous linear functionals on $L^\infty(G/H)$. There exist $f_i \in L^\infty(G/H)$, $i = 1, 2, \ldots, N$ such that $\nu_i(f_j) = \delta_{ij}$. It follows that

$$x \cdot \nu_i = \sum_{j=1}^{N} C_{ji}(x) \nu_j,$$

where

$$C_{ji}(x) = \nu_i((f_j)_{x^{-1}})$$

and the C_{ji} are clearly bounded, say

$$|C_{ji}(x)| \leqslant M, \qquad x \in G.$$

Let $|\nu_j|$ denote the variation of ν considered as a linear functional on $L^\infty(G/H)$ and let λ_j be the restriction of $|\nu_j|$ to $C_c(G/H)$.

In symbols, $\lambda_j = |\nu_j|\,|_{C_c(G/H)}$. Now

$$x \cdot \lambda_i = |x \cdot \nu_i|\,|_{C_c(G/H)} = \left|\sum C_{ji}(x)\nu_i\right|\,|_{C_c(G/H)}$$
$$\leqslant M \sum_j |\nu_j|\,|_{C_c(G/H)} = M \sum_j \lambda_j .$$

By Lemma 2, $\lambda_j = 0$ for all j. Therefore, $W \supseteq L_c^\infty(G/H) = L^\infty(G/H)$ since G/H is supposed compact, and the theorem is proved.

We mention that if we do not assume that G/H is compact, we may still conclude under the remaining hypotheses of Theorem 1 that $W \supseteq L_c^\infty(G/H)$. We remark that Theorem 1 implies that if G is a locally compact group for which there exists a closed subgroup H satisfying the hypotheses of Theorem 1, then G is not amenable. Consequently we have a new proof that a semisimple Lie group is not amenable.

We conclude with a problem. Suppose, under the hypotheses of Theorem 1 that E and F are closed invariant subspaces of $L^{\infty}(G/H)$, that $E \subseteq F$, $C(G/H) \subseteq F$, and that F is a module over $C(G/H)$. If $\dim F/E < \infty$, does it follow that $E = F$?

REFERENCES

1. N. BOURBAKI, "Eléments de Mathématiques, Intégration," Vol. 29, Livre VI, Paris.
2. F. P. GREENLEAF, "Invariant Means on Topological Groups," Van Nostrand, Reinhold, New York, 1969.
3. G. W. MACKEY, Induced representation of locally compact groups, *Ann. of Math.* **55** (1952), 101–139.
4. H. ROYDEN, "Real Analysis," Second ed., Macmillan, New York, 1968.
5. L. A. RUBEL AND A. L. SHIELDS, Invariant subspaces of L^{∞} and H^{∞}, *J. Reine Angew. Math.*, to appear; (research announcement) *Bull. Amer. Math. Soc.* **79** (1973), 136–137.
6. A. WEIL, "L'Intégration dans les Groupes Topologiques et ses Applications," Act. Sci. et Ind. no. 869, Hermann, Paris, 1938.